Computers Math. Applic. Vol. 24, No. 5/6, pp. 1–10, 1992
Printed in Great Britain.

0097-4943/92 $5.00 + 0.00

A NUMERICAL MODEL OF A THERMAL BOUNDARY LAYER

GARY A. SOD
Department of Mathematics, Tulane University
New Orleans, LA 70118, U.S.A.

Abstract — A grid-free method for approximating incompressible thermal boundary layers is introduced. The computational elements are segments of vortex sheets which are heated. The method solves the general Prandtl boundary layer equations. The method is applied to the parallel flow past a heated horizontal flat plate and a flow created entirely by buoyancy forces (free-convection flow) on a heated vertical flat plate.

1. INTRODUCTION

In [1] Chorin presented a grid-free numerical method that approximated the incompressible boundary layer equations in the absence of temperature variations. The algorithm approximated the vortex sheets within the boundary layer by segments of vortex sheets. The equations solved are the Prandtl boundary layer equations.

In this paper, we extend the vortex sheet algorithm to treat the more general incompressible boundary layer equations which allow for temperature variations and buoyancy effects (using the Boussinesq approximation). The equations of motion and energy are coupled by the buoyancy term in the momentum equation and by the advection and friction terms in the energy equation.

2. PRINCIPLE OF THE METHOD

The boundary layer equations can be written in the form (e.g., [2])

$$\partial_t u + (\underline{u} \cdot \underline{\nabla})\, u = \nu\, \partial y^2\, u - \frac{\partial_x p}{\rho} + g_x\, \beta\, (T - T_\infty)$$
$$\partial_x u + \partial_y v = 0$$
$$\partial_t T + (\underline{u} \cdot \underline{\nabla})\, T = k\, \partial y^2\, T + \left(\frac{\nu}{c_p}\right)\, (\partial_y u)^2$$

where $\underline{u} = (u, v)$ is the velocity, u is tangential to the boundary and v is normal to the boundary, x is the spatial coordinate tangential to the boundary and y is the spatial coordinate normal to the boundary, ρ is the density, p is the pressure, T is the temperature, ν is the kinematic viscosity, k is the thermal diffusivity, and c_p is the specific heat at constant pressure. The term $g_x\, \beta\, (T - T_\infty)$ represents a body force due to buoyancy forces caused by temperature differences, that is, thermal expansion, where g_x is the gravitational force in the x direction and β is the coefficient of thermal expansion.

We assume a wall is located at $y = 0$ and that the fluid occupies the half-space defined by $y \geq 0$. The boundary conditions are

$$\underline{u} = 0 \quad \text{at} \quad y = 0 \tag{1a}$$
$$u(x, \infty) = U_\infty(x) \tag{1b}$$

and

$$T = T_w \quad \text{at} \quad y = 0 \tag{2a}$$
$$T(x, \infty) = T_\infty(x). \tag{2b}$$

Introduce four dimensionless quantities, Re$= U_\infty L/\nu$, the Reynolds number, where L is a characteristic length, $\mathrm{Pr} = \nu/k$, the Prandtl number, $\mathrm{Ec} = U_\infty^2/c_p\,\Delta T$, the Eckert number, where $\Delta T = T_W - T_\infty$, and $\mathrm{Gr} = g\,\beta\,L^3\,\Delta T/\nu^2$, the Grashof number, where g is the gravitational constant. This gives rise to the dimensionless form of the equations

$$\begin{gathered}\partial_t u + (\underline{u}\cdot\underline{\nabla})\,u = \mathrm{Re}^{-1}\,\partial y^2\,u - \partial_x p + \mathrm{Gr}\,\mathrm{Re}^{-2}\,T,\\ \partial_x u + \partial_y v = 0,\\ \partial_t T + (\underline{u}\cdot\underline{\nabla})T = (\mathrm{Pr}\,\mathrm{Re})^{-1}\partial y^2 T + \mathrm{Ec}\,\mathrm{Re}^{-1}(\partial_y u)^2.\end{gathered} \tag{3}$$

Using operator splitting, we may write the energy equation as

$$\begin{gathered}\partial_t T + (\underline{u}\cdot\underline{\nabla})\,T = (\mathrm{Pr}\,\mathrm{Re})^{-1}\,\partial y^2\,T\\ \partial_t T = \mathrm{Ec}\,\mathrm{Re}^{-1}\,(\partial_y u)^2.\end{gathered} \tag{4}$$

Introduce the vorticity (within the boundary layer) $\xi = -\partial_y u$ and the heat flux $q = \partial_y T$. Taking the derivative with respect to y of Equations (3) and (4) we obtain

$$\partial_t\xi + (\underline{u}\cdot\underline{\nabla})\,\xi = \mathrm{Re}^{-1}\,\partial y^2\,\xi + \mathrm{Gr}\,\mathrm{Re}^{-2}\,q \tag{5}$$

$$\partial_x u + \partial_y v = 0 \tag{6}$$

$$\partial_t q + (\underline{u}\cdot\underline{\nabla})\,q \;=\; (\mathrm{Pr}\,\mathrm{Re})^{-1}\,\partial y^2\,q + q\,\partial_x u - \xi\,\partial_x T \tag{7a}$$

$$\partial_t T \;=\; \mathrm{Ec}\,\mathrm{Re}^{-1}\,\xi^2. \tag{7b}$$

Integration of the vorticity field ξ yields

$$u(x,y) = U_\infty(x) - \int_y^\infty \xi(x,s)\,ds \tag{8}$$

Using this along with the Continuity Equation (6) we obtain

$$\begin{aligned} v(x,y) &= -\partial_x \int_0^y u(x,z)\,dz\\ &= -\partial_x \int_0^y \left[U_\infty(x) - \int_z^\infty u(x,s)\,ds\right]\,dz\\ &= -\partial_x \left[U_\infty(x)y - \int_0^y \int_z^\infty u(x,s)\,ds\,dz\right]\\ &= -\partial_x \left[U_\infty(x)y - \int_0^\infty \int_0^{\min(s,y)} \xi(x,s)\,dz\,ds\right]\\ &= -\partial_x \left[U_\infty(x)y - \int_0^\infty \xi(x,s)\,\min(s,y)\,ds\right]. \end{aligned} \tag{9}$$

Thus, if the vorticity field ξ is known, the velocity field $\underline{u}$ can be determined.

Similarly, we may integrate heat flux q to obtain

$$T(x,y) = T_\infty(x) - \int_y^\infty q(x,s)\,ds. \tag{10}$$

Consider a collection of N segments of vortex sheets with intensities of ξ_i and heat fluxes of q_i, $i = 1,\ldots,N$. The S_i are straight line segments parallel to the x-axis having equal length h and center (x_i, y_i). The intensity and heat flux are uniformly distributed over the length of the segment S_i.

The difference in u across the segment S_i is ξ_i, that is,

$$\xi_i = -(u(x_i, y_i^+) - u(x_i, y_i^-))$$

and the difference in T across the segment S_i is q_i, that is,

$$q_i = T(x_i, y_i^+) - T(x_i, y_i^-).$$

Let

$$d_j = \begin{cases} 1 - |x_i - x_j|/h, & \text{if } |x_i - x_j| < h \\ 0, & \text{otherwise,} \end{cases}$$

then

$$u_i = U_\infty - \frac{1}{2}\xi_i - \sum_{\substack{j \neq i \\ y_j > y_i}} \xi_j \, d_j \tag{11}$$

and

$$T_i = T_\infty - \sum_{\substack{j \neq i \\ y_j > y_i}} q_j d_j . = \tag{12}$$

The number of interactions among the segments is small since for a segment S_j to influence a given segment S_i it must intersect the semi-infinite strip $R_i = \{(x, y) \mid x_i - h/2 \leq x \leq x_i + h/2\}$, where $x_i - h/2$ and $x_i + h/2$ are the ends of the segment S_i.

This term d_j is a weight which corresponds to the length of the portion on of the segment S_j which lies in the strip R_i. If S_j does not intersect the strip R_i then $d_j = 0$.

Below the sheet S_i, the velocity is increased by ξ_i and the temperature is increased by q_i. The velocity of a segment at a height y is increased by all segments of sheets which lie above y. Similarly, the temperature of a segment at a height y is increased by all segments of sheets which lie above y.

Use a centered difference approximation to ∂_x to obtain

$$\begin{aligned} &-\frac{1}{h}\left[\int_0^{y_i} u\left(x_i + \frac{h}{2}, y\right) dy - \int_0^{y_i} u\left(x_1 - \frac{h}{2}, y\right) dy\right] \\ &= -\frac{1}{h}\Bigg[\left(U_\infty\left(x_i + \frac{h}{2}\right) - U_\infty\left(x_i - \frac{h}{2}\right)\right) y_i \\ &\quad - \left(\int_0^\infty \xi\left(x_i + \frac{h}{2}, s\right) \min(s, y_i)\, ds - \int_0^\infty \xi\left(x_i - \frac{h}{2}, s\right) \min(s, y_i)\, ds\right)\Bigg]. \end{aligned}$$

The integral $\int_0^{y_i} u(x_i + h/2, y)dy$ is the flux of u across a segment of the vertical line at $x_i + h/2$ between 0 and y_i. Each segment which intersects this vertical line segment affects the velocity of this line and hence the flux. If a segment S_j lies above y_i, that is, $y_j > y_i$, then S_j increases the velocity of the entire vertical segment from 0 to y_i and hence the flux is increased by $y_i\, \xi_j$. If a segment S_j lies below y_i, that is, $y_j < y_h$, then S_j increases the velocity of the vertical line segment by ξ_j below S_j and hence increases the flux by $y_j\, \xi_j$. This is summarized as follows

$$v_i = -\frac{1}{h}\left[\left(U_\infty\left(x_i + \frac{h}{2}\right) - U_\infty\left(x_i - \frac{h}{2}\right)\right) y_i - \left(\sum_{j \neq i} \xi_j \, d_j^+ \, y_j^{\min} - \sum_{j \neq i} \xi_j \, d_j^- \, y_j^{\min}\right)\right] \tag{13}$$

where

$$d_j^\pm = \begin{cases} 1 - |x_i \pm \frac{h}{2} - x_j|/h, & \text{if } |x_i \pm \frac{h}{2} - x_j| < h \\ 0, & \text{otherwise} \end{cases}$$

and

$$y_j^{\min} = \min(y_i, y_j).$$

We now describe the algorithm. Operator splitting is used to write Equations (5)–(7) in terms of advection, diffusion and source terms. The advection process, described by

$$\begin{aligned}
&\partial_t \xi + (\underline{u} \cdot \underline{\nabla})\, \xi = 0, \\
&\partial_t q + (\underline{u} \cdot \underline{\nabla})\, q = 0, \\
&\xi = -\partial_y u, \\
&q = \partial_y T, \\
&\partial_x u + \partial_y v = 0,
\end{aligned}$$

can be approximated by

$$\begin{aligned}
x_i^{n+1} &= x_i^n + \Delta t\, u_i^n, \\
y_i^{n+1} &= y_i^n + \Delta t\, v_i^n,
\end{aligned}$$

where u_i^n and v_i^n are determined using (11) and (13) and Δt denotes the time step.

The diffusion process shall be approximated by a random walk of the segments. However, Equations (5) and (7a) may have different diffusion coefficients, while the heat flux and vorticity are assigned to the same sheets. In the case where $\mathrm{Pr} = 1$, no further diffusion takes place. This problem can be remedied by splitting the diffusion term in (5) or (7a) into the sum of two diffusion terms

$$\begin{aligned}
\partial_t q &= (\mathrm{Pr}\,\mathrm{Re})^{-1}\, \partial_y^2 q = \mathrm{Re}^{-1}\, \partial_y^2 q + (1-\mathrm{Pr})(\mathrm{Pr}\,\mathrm{Re})^{-1}\, \partial_y^2 q, \qquad \text{if } \mathrm{Pr} < 1, \\
\partial_t \xi &= \mathrm{Re}^{-1} \partial_y^2 \xi = (\mathrm{Pr}\,\mathrm{Re})^{-1}\, \partial_y^2 \xi + (\mathrm{Pr}-1)(\mathrm{Pr}\,\mathrm{Re})^{-1}\, \partial_y^2 \xi, \qquad \text{if } \mathrm{Pr} > 1.
\end{aligned} \tag{14}$$

Consider, without a loss of generality, the case where $Pr < 1$,

$$\begin{aligned}
&\partial_t \xi = \mathrm{Re}^{-1}\, \partial_y^2 \xi, \\
&\partial_t q = \mathrm{Re}^{-1}\, \partial_y^2 q.
\end{aligned}$$

These can be solved deterministically by choosing a random variable η_i drawn from a Gaussian distribution with mean 0 and variance $2\Delta t/\mathrm{Re}$. This yields the algorithm

$$\begin{aligned}
x_i^{n+1} &= x_i^n + \Delta t\, u_i^n, \\
y_i^{n+1} &= y_i^n + \Delta t\, v_i^n + \eta_i.
\end{aligned}$$

The remaining diffusion term

$$\partial_t q = (1-\mathrm{Pr})(\mathrm{Pr}\,\mathrm{Re})^{-1} \partial_y^2 q \tag{15}$$

is treated by a separate random walk. Define the new flux $q_f = \partial_y q$. Assume that $q \to 0$ as $y \to \infty$, then

$$q = \int_y^\infty q_f\, dy. \tag{16}$$

Differentiating Equation (15) with respect to y gives

$$\partial_t q_f = (1-\mathrm{Pr})(\mathrm{Pr}\,\mathrm{Re})^{-1} \partial_y^2 q_f. \tag{17}$$

Divide the domain Ω into a grid of spacing h in the x direction and an increment Δy in the y direction. The node (r, s) corresponds to the point $\overline{x}_r = rh$ and $\overline{y}_s = s\Delta y$. The heat flux q_i from the N sheets must be spread onto the grid. Consider the distribution of heat flux on the grid

$$q(\underline{x}_{r,s}) = \iint_\Omega q(\underline{x})\, \delta(\underline{x} - \underline{x}_{r,s})\, d\underline{x}, \tag{18}$$

where δ denotes the Dirac delta function, $\underline{x} = (x, y)$, and $\underline{x}_{r,s} = (x_r, y_s)$. Define the smooth approximation to $\delta(\underline{x})$ by

$$d(\underline{x}) = \begin{cases} \frac{1}{16h\Delta y(1+\cos(\frac{\pi x}{2h}))(1+\cos(\frac{\pi y}{2\Delta y}))}, & |x| < 2h \text{ and } |y| < 2\Delta y \\ 0, & \text{otherwise.} \end{cases}$$

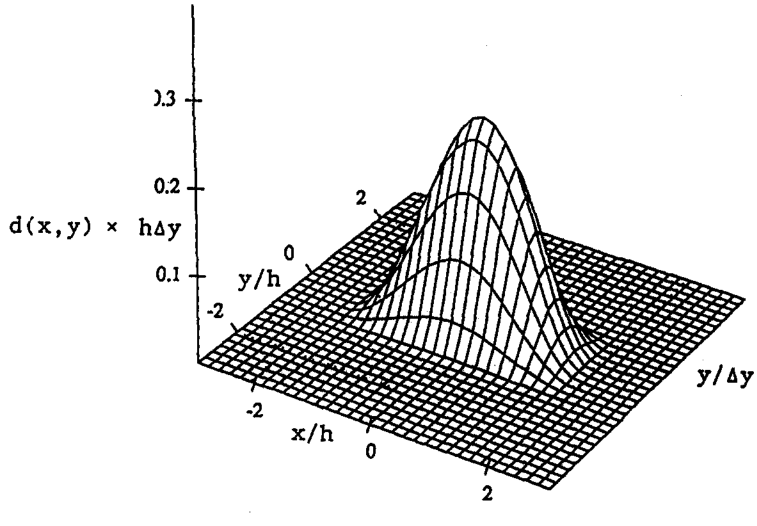

Figure 1a. Approximate delta function d($\underline{x}$).

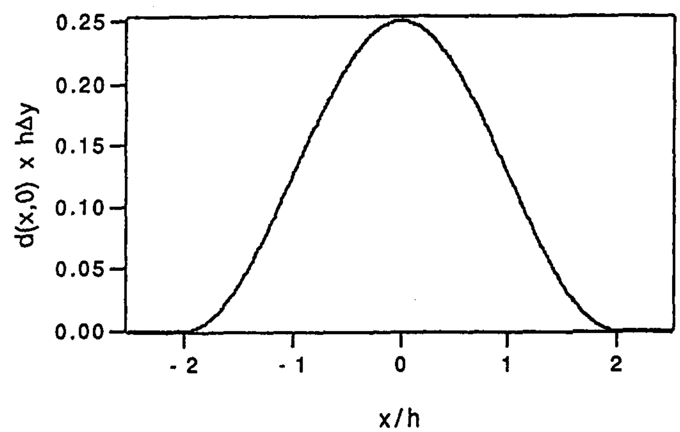

Figure 1b. Boundary Layer of a parallel flow past a heated horizontal plane.

The shape of $d(\underline{x})$ is shown in Figure 1a. The cross section of $d(\underline{x})$ corresponding to the plane $y = 0$ is shown in Figure 1b. Then $d(\underline{x})$ has compact support and satisfies

$$\iint_{\mathbf{R}^2} d(\underline{x})\, d\underline{x} = \sum_{r,s} d(\underline{x} - \underline{x}_{r,s})\, h\, \Delta y = 1.$$

We define the discrete approximation to (18),

$$q_{r,s} = \sum_{j=1}^{N} q_j d(\underline{x}_j - \underline{x}_{r,s}) h \Delta y,$$

where $\underline{x}_j = (x_j, y_j)$. The quantity $q_{r,s}$ denotes the heat flux at the node (r, s). Property (19) assures that even in the discrete case, the total heat flux of the N sheets is conserved when spread onto the grid.

Define the flux at the node (r, s)

$$(q_f)_{rs} = q_{rs} - q_{r,s-1}.$$

At each node (r, s) divide the flux $(q_f)_{rs}$ among N_{rs} tiles of length h and center (r, s) with mass $(q_f)_{rs}/N_{rs}$, where N_{rs} is chosen so that $|(q_f)_{rs}|/N_{rs} < q_f^{\max}$. There are a total of $N_f = \sum_{r,s} N_{rs}$ tiles. Giving these tiles a single index, the tiles have center (x_{f_i}, y_{f_i}) which corresponds to

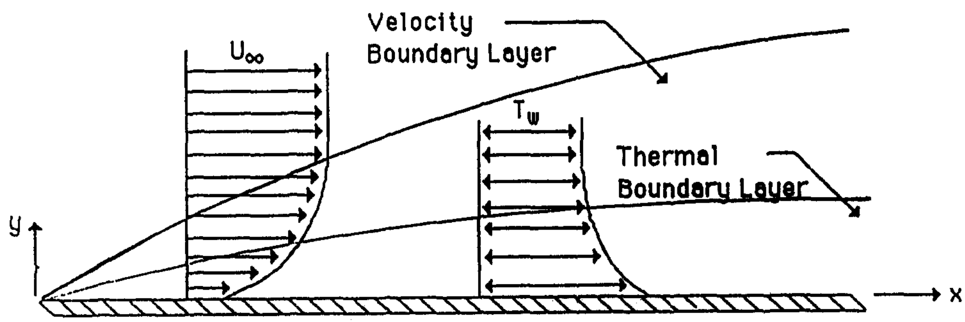

Figure 2. Temperature profiles in a forced convection boundary layer on a heated ($E_c \geq 0$) and a cooled ($E_c < 0$) horizontal flat plate in a parallel stream.

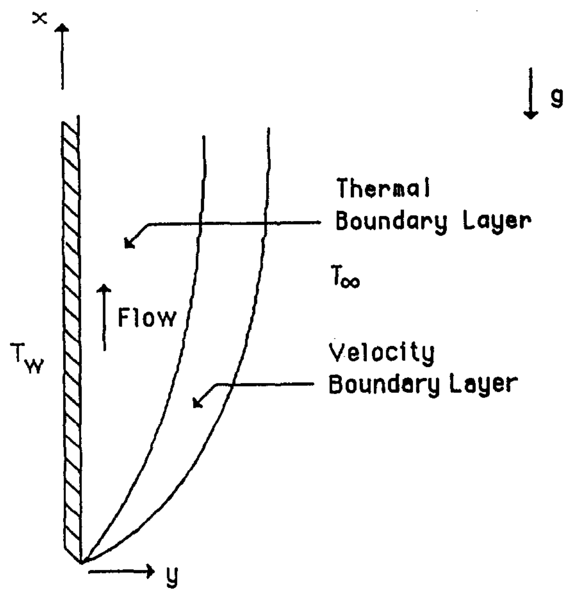

Figure 3. Temperature profiles in a forced convection boundary layer on a heated ($E_c \geq 0$)

the nodes (r, s), and strength $q_{f_i}, i = 1, \ldots, N^f$. To solve Equation (17), let each tile undergo a random walk in the y direction so that the new location of the i^{th} tile is $(x_{f_i}, \overline{y}_{f_i})$ where $\overline{y}_{f_i} = y_{f_i} + \eta_{f_i}$, and η_{f_i} drawn from a Gaussian distribution with mean 0 and variance $2\Delta t\,(1 - \mathrm{Pr})(\mathrm{Pr}\,\mathrm{Re})$. These tiles have now left the grid. The effect of the diffusion process on the vortex sheets is realized by computing new values of q_i using (16),

$$q_i = \int_{y_i}^{\infty} q_f \, dy$$

which is approximated in a manner similar to T_i in (10), that is,

$$q_i = \sum_{\substack{j=1 \\ \overline{y}_{f_j} > y_i}}^{N_f} q_{f_j} \, d_{f_j}$$

where

$$d_{f_j} = \begin{cases} 1 - \frac{|x_i - x_{f_j}|}{h}, & \text{if } |x_i - x_{f_j}| < h \\ 0, & \text{otherwise.} \end{cases}$$

It remains to describe the treatment of the source terms in (5) and (7). From (5)

$$\partial_t \xi = \mathrm{Gr}\,\mathrm{Re}^{-2}\, q$$

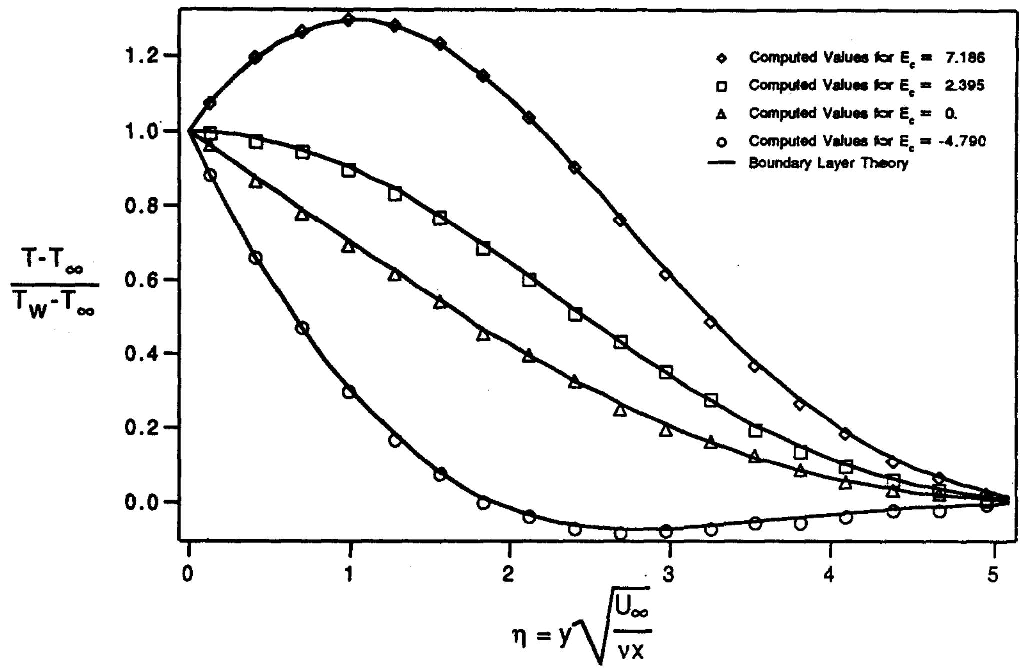

Figure 4. Boundary layer of a natural convection flow a vertical plate with $T_w > T_\infty$.

is solved to modify the intensity ξ_i of the N sheets,

$$\xi_i^{n+1} = \xi_i^n + \Delta t \, \mathrm{Gr}\mathrm{Re}^{-2} \, q_i^n,$$

using Euler's method. From Equation (7a), ignoring the friction term,

$$\partial_t q = q \, \partial_x u - \xi \, \partial_x T$$

is solved using Euler's method to modify the heat flux q_i of the N sheets,

$$\begin{aligned} q_i^{n+1} = q_i^n \; &+ \; \Delta t \, q_i^n \, \frac{u^n\left(x_i + \frac{h}{2}, y_i\right) - u^n\left(x_i - \frac{h}{2}, y_i\right)}{h} \\ &- \; \Delta t \, \xi_i^n \, \frac{T^n\left(x_i + \frac{h}{2}, y_i\right) - T^n\left(x_i - \frac{h}{2}, y_i\right)}{h} \end{aligned}$$

To treat frictional effects on temperature, we consider Equation (7b),

$$\partial_t T = \mathrm{Ec} \, \mathrm{Re}^{-1} \, \xi^2.$$

This can be solved using Euler's method,

$$T_i^{n+1} = T_i^n + \Delta t \, \mathrm{Ec} \, \mathrm{Re}^{-1} \, (\xi_i^n)^2. \tag{19}$$

We now must use this to modify the heat flux of the sheets. First observe that $T_i^{n+1} \equiv T(x_i, y_i^-)$. To see this, we consider (12) where y is so large that there are no sheets above it then $T = T_\infty$, a constant. Suppose the N sheets are ordered so that $y_1 \geq y_2 \geq y_3 \geq \cdots \geq y_N$. Then $T(x, y_1^+) = T_\infty$ and $T(x, y) = T_\infty - q_1 d_1$ for $y_2 < y < y_1$. Also $\lim\limits_{y \to y_1^-} T(x, y) = T_\infty - q_1 d_1 = T(x, y_1^-)$. Similar results hold for $T(x, y_i^-)$. The algorithm is inspired by this observation. With the ordering $y_1 \geq y_2 \geq \cdots \geq y_N$,

$$q_i^{n+1} = T^{n+1}(x_i, y_i^+) - T_i^{n+1}(x_i, y_i^-)$$

where $T^{n+1}(x_i, y_i^-)$ is given by (20) and

$$T^{n+1}(x_i, y_i^+) = T_\infty - \sum_{j=1}^{i-1} q_j^{n+1} \, d_j.$$

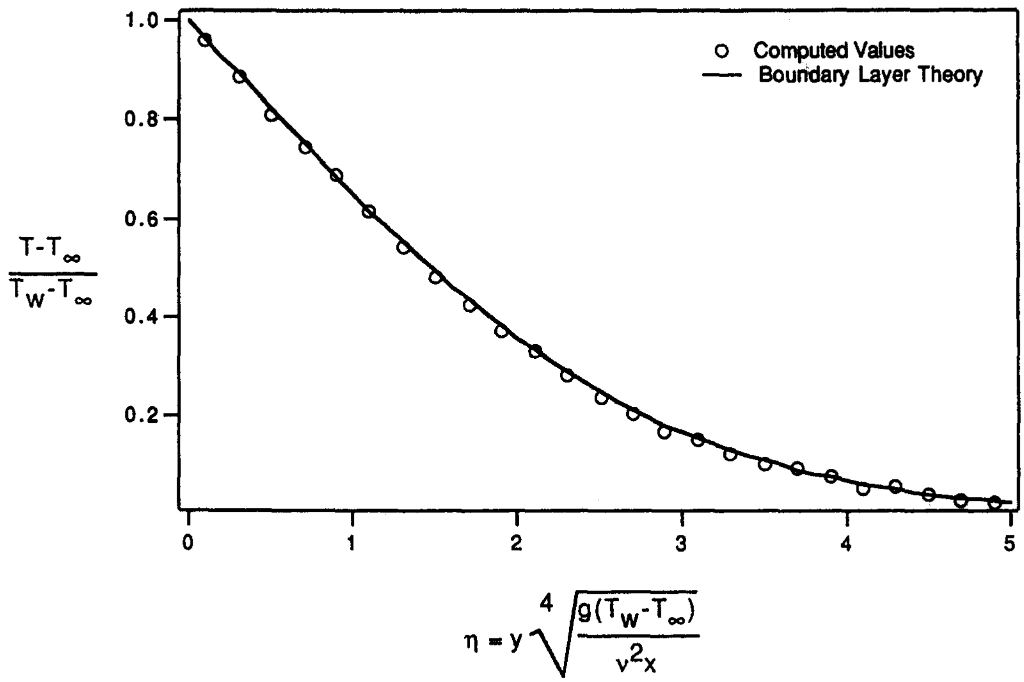

Figure 5. Temperature profile in a natural convection boundary layer on a heated vertical flat plate.

It should be noted that the ordering of the sheets by a sorting algorithm will also minimize the number of decisions involved in carrying out the various summations.

Vorticity and Heat Flux Creation

Using (11) we see that boundary condition (1b) is satisfied and (13) we see that the normal component on the "no-slip" condition (1a) is satisfied. Similarly using (12) we see that boundary condition (2b) is satisfied. However, the tangential component of the "no-slip" condition (1a) and the boundary condition (2a) are not satisfied.

Divide the boundary into segments of length h with centers $(\overline{x}_i, 0)$. Using (11) we can compute

$$u(\overline{x}_i, 0^+) = U_\infty - \sum_j \xi_j \, d_j.$$

The "no slip" condition requires $u(\overline{x}_i, 0^-) = 0$. So a vortex sheet must be created at $(\overline{x}_i, 0)$ of strength $\overline{\xi}_i = u(\overline{x}_i, 0^+) - u(\overline{x}_i, 0^-) = u(\overline{x}_i, 0^+)$ per unit length.

Similarly, using (10)

$$T(\overline{x}_i, 0^+) = T_\infty - \sum_j q_j \, d_j$$

while the boundary condition (2a) requires that $T(\overline{x}_i, 0^-) = T_W$. So the heat flux of the vortex sheet created at $(x_i, 0)$ is $\overline{q}_i = T(\overline{x}_i, 0^+) - T(\overline{x}_i, 0^-) = T(\overline{x}_i, 0^+) - T_W$ per unit length.

The vortex sheet at $(\overline{x}_i, 0)$ is broken up into m_i elements, each having intensity $\overline{\xi}_i/m_i$ and heat flux $\overline{q}_i/m_i$ where m_i is chosen so that $|\overline{q}_i|/m_i \le q^{\max}$ and $|\overline{\xi}_i|/m_i \le \xi^{\max}$ simultaneously.

Consider the case in which $U_\infty = 0$, that is, free-convection flow, so that $\xi_i = 0$ for the sheets. Solving the ordinary differential equation stemming from the source term in (5)

$$\partial_t \xi = \text{Gr Re}^{-2} T,$$

results in the production of vorticity. Suppose $\xi_i^u = 0$ then $\xi_i^{n+1} = \Delta t \text{Gr Re}^{-2} T_i^n$ is the strength of the i^{th} sheet. If $\xi_i^{n+1} > \xi^{\max}$ then the sheet is subdivided into some number of sheets each

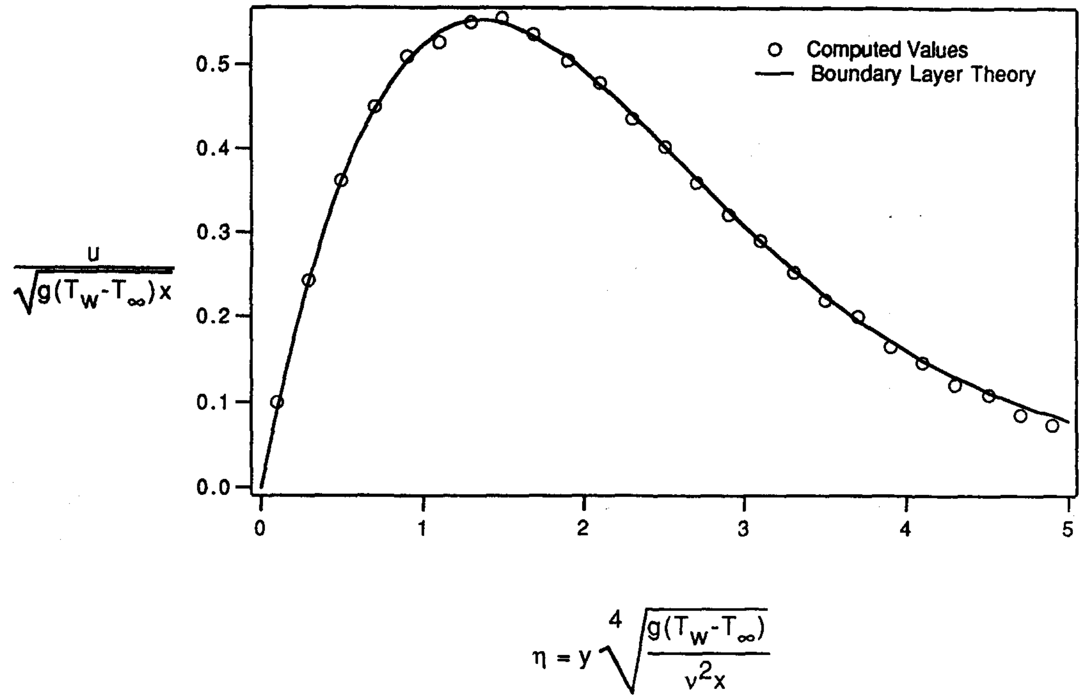

Figure 6. Velocity profile in a natural convection boundary layer on a heated vertical flat plate.

with center (x_i, y_i) and strength less than $\xi^{\max}$ and sum of strengths equal to ξ_i^{n+1}. The heat flux q_i^n is divided equally among these sheets so that the total heat flux has not been altered. Each of these subdivided sheets will undergo a random walk and will each have different locations at subsequent times.

3. APPLICATION TO FLOW PAST A HEATED FLAT PLATE

Consider a semi-infinite flat plate located on the positive x-axis. The fluid at temperature T_∞ occupies the half plane $y > 0$. At time $t = 0$ the plate is instantaneously and uniformly heated to temperature T_w and the fluid is impulsively set in motion with velocity U_∞ (see Figure 2). We shall compare the results of our method to known solutions (see [2]).

Heat due to friction is important only if $\mathrm{Ec} = O(1)$. In the case of a gas, this is equivalent to requiring that the temperature rise due to adiabatic compression be of the same order of magnitude as the difference in temperature between the plate and the fluid.

We consider four cases, each with $Pr = .71$ (air), $G_r = 0$ (buoyancy neglected), $U_\infty = 1$, and $L = 1$. In the first two cases, we consider a heated flat plate when the effects of frictional heating are important, that is, $E_c > 0$. In the third case, frictional heating is negligible, that is, $E_c = 0$. And finally, in the fourth case, we consider a cooled flat plate when the effects of frictional heating are important.

In the first case, $E_c = 7.186$ with $\mathrm{Re} = 4.56 \times 10^6, T_w = 1$ and $T_\infty = 0$. In the second case, corresponding to an adiabatic wall, $E_c = 2.395$ with Re$= 2.4 \times 10^6, T_w = 1$ and $T_\infty = 0$. In the third case, $E_c = 0$ with $\mathrm{Re} = 5.4 \times 10^5$, $T_w = 1$ and $T_\infty = 0$. In the fourth case, $E_c = -4.790$ with $\mathrm{Re} = 3.7 \times 10^6$, $T_w = 0$ and $T_\infty = 1$.

The numerical parameters are $\Delta t = .2$, $h = .2$, $\xi^{\max} = .1$, $q^{\max} = .1$. In Figure 3, we display the temperature profiles for all four cases averaged over 10 time-steps for $16 < t \leq 18$. The results are in excellent agreement with the similarity solutions superimposed on the figures.

For $E_c > 2.395$, the boundary layer near the plate is hotter than the plate itself. The generation of frictional heat provides a "heat cushion" which prevents the cooling of the heated plate by the stream of cooler air. This effect can be seen in Figure 3 for the first case where $E_c = 7.186$. The region near the local maximum is the "heat cushion".

4. APPLICATION TO NATURAL-CONVECTION BOUNDARY LAYERS

Consider a semi-infinite vertical flat plate placed in a fluid at rest. Assume the coordinate system is oriented in such a way that the plate lies along the x-axis (see Figure 4). The fluid at temperature $T_\infty = 0$ occupies the half plane $\eta = 0$. Assume the plate is heated to a temperature $T_w > T_\infty$, in which case the fluid adjacent to the plate is heated, becomes lighter, and rises, resulting in the formation of a velocity and thermal boundary layers (see Figure 4).

There is no free stream, so there is no reference velocity. One possible reference velocity is

$$U_\infty = \sqrt{g\,\beta(T_w - T_\infty)\,L},$$

which contains the buoyant terms. With this reference velocity we can define the Reynolds number

$$\mathrm{Re} = \frac{U_\infty\,L}{\nu} = \frac{\sqrt{g\,\beta(T_w - T_\infty)}\,L^{3/2}}{\nu} = G_r^{1/2}$$

so that Gr $\mathrm{Re}^{-2} = 1$.

We consider the case when friction is negligible, that is, $E_c = 0$. The parameters are $\mathrm{Pr} = .71$, $\mathrm{Re} = 1.68 \times 10^5, T_w = 1, T_\infty = 0$, and $L = 1$. The numerical paramters are $\Delta t = .2$, $h = .2$, $\xi^{\max} = .1$, $q^{\max} = .1$. In Figures 5 and 6, we display the velocity and temperature profiles, respectively, averaged over 10 time-steps for $16 < t \leq 18$. The results are in excellent agreement with the similarity solutions superimposed on the figures.

REFERENCES

1. A. J. Chorin, Vortex sheet approximations of boundary layers, *J. Comp. Phys.* **27**, page 428 (1978).
2. H. Schlichting, *Boundary-Layer Theory*, 6th. Edition, McGraw-Hill Book Company (1968).